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MULTIVALUED DISPLACEMENTS AND VOLTERRA DISLOCATIONS IN PLANE NONLINEAR ELASTICITY THEORY

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The problem is solved of determining the plane displacement field of a continuous medium by means of a unique finite strain tensor field given in a non-simply connected plane domain and satisfying the nonlinear compatibility equation. Given for the plane problem is a generalization of the classical Weingarten theorem to the case of large strains. An expression is obtained for the Burgers and Frank vectors of the Volterra dislocation (isolated defect) in terms of the finite strain tensor field. Given is a formulation of the plane problem of determining the stresses in a nonlinearly elastic body containing an isolated defect with given characteristics. An exact solution of the problems of a wedge disclination is found for a specific model of a nonlinearly elastic material. It is established that the stress field has no singularities on the disclination axis for a nonlinear formulation of the problem.

1. The plane strain of a continuous medium is described by the relationships

$$X_1 = X_1(x_1, x_2),\ X_2 = X_2(x_1, x_2),\ X_3 = x_3, \tag{1.1}$$

where x_k and X_k are Cartesian coordinates of points of the medium, respectively, before and after strain. We denote the coordinate directions by $\mathbf{e}_k$ (k = 1, 2, 3). We introduce complex coordinates and their associated vector bases [1-5]

$$\zeta = x_1 + ix_2,\ \bar\zeta = x_1 - ix_2,\ z = X_1 + iX_2,\ \bar z = X_1 - iX_2,$$
$$\mathbf{f}_1 = \bar{\mathbf{f}}_2 = \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2),\quad \mathbf{f}^1 = \bar{\mathbf{f}}^2 = \mathbf{e}_1 + i\mathbf{e}_2,\quad \mathbf{f}^3 = \mathbf{f}_3 = \mathbf{e}_3,\quad \mathbf{f}_k.\mathbf{f}^n = \delta_k^n.$$

Here δ_k^n is the Kronecker delta. The plane strain (1.1) can evidently be given by using the complex-valued function

$$z = z(\zeta, \bar\zeta),\ X_3 = x_3. \tag{1.2}$$

The site gradient (distortion tensor) [2, 6] corresponding to the transformation (1.2) has the form

$$\mathbf{C} = \frac{\partial X_k}{\partial x_n}\mathbf{e}_n\mathbf{e}_k = \frac{\partial z}{\partial \zeta}\mathbf{f}^1\mathbf{f}_1 + \frac{\partial \bar z}{\partial \zeta}\mathbf{f}^1\mathbf{f}_2 + \frac{\partial z}{\partial \bar\zeta}\mathbf{f}^2\mathbf{f}_1 + \frac{\partial \bar z}{\partial \bar\zeta}\mathbf{f}^2\mathbf{f}_2 + \mathbf{f}^3\mathbf{f}_3. \tag{1.3}$$

A polar expansion of the site gradient results [7, pp. 59, 60] in the measure of the distortion $\mathbf{U}$ which is a symmetric positive-definite tensor of the second rank, and to an intrinsically orthogonal rotation tensor $\mathbf{A}$

$$\mathbf{C} = \mathbf{U}\cdot\mathbf{A},\ \mathbf{U} = \mathbf{G}^{1/2},\ \mathbf{G} = \mathbf{C}\cdot\mathbf{C}^T. \tag{1.4}$$

The Cauchy–Green finite strain tensor $\mathbf{E}$ is expressed in terms of the Cauchy strain measure $\mathbf{G}$ by the relationship [2, p. 24]

$$\mathbf{E} = (1/2)(\mathbf{G} - \mathbf{I}) \tag{1.5}$$

($\mathbf{I}$ is the unit tensor). For plane strain the rotation tensor has the representation

$$\mathbf{A} = (\mathbf{I} - \mathbf{e}_3\mathbf{e}_3)\cos\chi + (\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1)\sin\chi + \mathbf{e}_3\mathbf{e}_3 = e^{i\chi}\mathbf{f}^1\mathbf{f}_1 + e^{-i\chi}\mathbf{f}^2\mathbf{f}_2 + \mathbf{f}^3\mathbf{f}_3, \tag{1.6}$$

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where χ is the angle of rotation of the principal strain axes. We find the Cauchy strain measure from (1.3)

$$\mathbf{G} = G_1^1\mathbf{f}^1\mathbf{f}_1 + G_1^2\mathbf{f}^1\mathbf{f}_2 + G_2^1\mathbf{f}^2\mathbf{f}_1 + G_2^2\mathbf{f}^2\mathbf{f}_2 + \mathbf{f}^3\mathbf{f}_3,$$
$$G_1^1 = G_2^2 = \frac{\partial \bar z}{\partial \bar\zeta}\frac{\partial z}{\partial \bar\zeta} + \frac{\partial \bar z}{\partial \zeta}\frac{\partial z}{\partial \zeta}, \quad G_1^2 = \bar G_2^1 = 2\frac{\partial z}{\partial \zeta}\frac{\partial \bar z}{\partial \zeta}. \tag{1.7}$$

The complex components G_α^β of the tensor $\mathbf{G}$ are expressed in terms of its Cartesian components $G_{\alpha\beta} = \mathbf{e}_\alpha\cdot\mathbf{G}\cdot\mathbf{e}_\beta$ by means of the formulas $G_1^1 = \frac{1}{2}(G_{11}+G_{22})$, $G_1^2 = \frac{1}{2}(G_{11}-G_{22}-2iG_{12})$.

Let us formulate the problem of determining the plane displacement field by means of a given tensor $\mathbf{E}(x_1, x_2)$. According to (1.5), this problem is equivalent to the problem of finding the function $z(\zeta, \bar\zeta)$ from the nonlinear system of equations (1.7) for given continuously differentiable functions $G_\alpha^\beta(\zeta, \bar\zeta)$. In the case of plane strain the compatibility equations in $G_{\alpha\beta}$ reduce to one relationship that denotes the disappearance of the component R_{1212} of the Riemann–Christoffel curvature tensor constructed in the $G_{\alpha\beta}$ metric:

$$\begin{aligned}(G_{11}G_{22} - G_{12}^2)\left(\frac{\partial^2 G_{22}}{\partial x_1^2} - 2\frac{\partial^2 G_{12}}{\partial x_1\partial x_2} + \frac{\partial^2 G_{11}}{\partial x_2^2}\right) + G_{11}\left(\frac{\partial G_{12}}{\partial x_1}\frac{\partial G_{22}}{\partial x_2} - \right.\\ \left. -\frac{1}{2}\frac{\partial G_{11}}{\partial x_2}\frac{\partial G_{22}}{\partial x_2} - \frac{1}{2}\left(\frac{\partial G_{22}}{\partial x_1}\right)^2\right) + G_{22}\left(\frac{\partial G_{11}}{\partial x_1}\frac{\partial G_{12}}{\partial x_2} - \frac{1}{2}\frac{\partial G_{11}}{\partial x_1}\frac{\partial G_{22}}{\partial x_1} - \right.\\ \left. -\frac{1}{2}\left(\frac{\partial G_{11}}{\partial x_2}\right)^2\right) - G_{12}\left(2\frac{\partial G_{12}}{\partial x_1}\frac{\partial G_{12}}{\partial x_2} - \frac{\partial G_{12}}{\partial x_1}\frac{\partial G_{22}}{\partial x_1} - \frac{\partial G_{11}}{\partial x_2}\frac{\partial G_{12}}{\partial x_2} + \right.\\ \left. + \frac{1}{2}\frac{\partial G_{11}}{\partial x_1}\frac{\partial G_{22}}{\partial x_2} - \frac{1}{2}\frac{\partial G_{11}}{\partial x_2}\frac{\partial G_{22}}{\partial x_1}\right) = 0.\end{aligned} \tag{1.8}$$

In complex variables this relationship is written in the form

$$\begin{aligned}(G_1^1G_2^2 - G_2^1G_1^2)\left(2\frac{\partial^2 G_1^1}{\partial\zeta\partial\bar\zeta} - \frac{\partial^2 G_1^2}{\partial\bar\zeta^2} - \frac{\partial^2 G_2^1}{\partial\zeta^2}\right) + G_1^2\left(\frac{\partial G_1^1}{\partial\zeta}\frac{\partial G_2^1}{\partial\bar\zeta} - \right.\\ \left. -\frac{1}{2}\frac{\partial G_1^2}{\partial\bar\zeta}\frac{\partial G_2^1}{\partial\bar\zeta} - \frac{1}{2}\left(\frac{\partial G_2^1}{\partial\zeta}\right)^2\right) + G_2^1\left(\frac{\partial G_1^1}{\partial\bar\zeta}\frac{\partial G_1^2}{\partial\zeta} - \frac{1}{2}\frac{\partial G_1^2}{\partial\zeta}\frac{\partial G_2^1}{\partial\zeta} - \right.\\ \left. -\frac{1}{2}\left(\frac{\partial G_1^2}{\partial\bar\zeta}\right)^2\right) - G_1^1\left(2\frac{\partial G_1^1}{\partial\zeta}\frac{\partial G_1^1}{\partial\bar\zeta} - \frac{\partial G_1^1}{\partial\bar\zeta}\frac{\partial G_1^2}{\partial\bar\zeta} - \frac{\partial G_1^1}{\partial\zeta}\frac{\partial G_2^1}{\partial\zeta} + \right.\\ \left. + \frac{1}{2}\frac{\partial G_1^2}{\partial\zeta}\frac{\partial G_2^1}{\partial\bar\zeta} - \frac{1}{2}\frac{\partial G_1^2}{\partial\bar\zeta}\frac{\partial G_2^1}{\partial\zeta}\right) = 0.\end{aligned} \tag{1.9}$$

On the basis of (1.4)

$$\mathbf{C} = U_1^1 e^{i\chi}\mathbf{f}^1\mathbf{f}_1 + U_1^2 e^{-i\chi}\mathbf{f}^1\mathbf{f}_2 + U_2^1 e^{i\chi}\mathbf{f}^2\mathbf{f}_1 + U_2^2 e^{-i\chi}\mathbf{f}^2\mathbf{f}_2 + \mathbf{f}^3\mathbf{f}_3,$$
$$U_\alpha^\beta = \mathbf{f}_\alpha\cdot\mathbf{U}\cdot\mathbf{f}^\beta. \tag{1.10}$$

Comparing the expressions (1.3) and (1.10), we obtain

$$\frac{\partial z}{\partial \zeta} = U_1^1 e^{i\chi}, \quad \frac{\partial z}{\partial \bar\zeta} = U_2^1 e^{i\chi}. \tag{1.11}$$

Using the formulas presented in [8, p. 64], we write explicit expressions for the components of the distortion measure in terms of components of the tensor $\mathbf{G}$:

$$U_1^1 = U_2^2 = \frac{1}{2}\sqrt{2\sqrt{G_1^1G_2^2 - G_1^2G_2^1} + 2G_1^1},$$
$$U_1^2 = (\bar U_2^1) = \frac{1}{2}(U_1^1)^{-1}G_1^2. \tag{1.12}$$

If the field of rotations $\chi(\zeta, \bar\zeta)$ were known then according to (1.11) the function $z(\zeta, \bar\zeta)$ would be determined by the quadrature $z = \int e^{i\chi}(U_1^1 d\zeta + U_2^1 d\bar\zeta)$.

The condition for integrability of the system (1.11) for $z(\zeta, \bar\zeta)$ evidently has the form

$$\frac{\partial}{\partial\bar\zeta}(U_1^1 e^{i\chi}) = \frac{\partial}{\partial\zeta}(U_2^1 e^{i\chi}). \tag{1.13}$$

The relationship (1.13) and its complex conjugate comprise equations to determine the field of rotations. Indeed, the mentioned relationships are transformed as follows

$$\frac{\partial\chi}{\partial\zeta} = \eta(\zeta, \bar{\zeta}), \quad \frac{\partial\chi}{\partial\bar{\zeta}} = \overline{\eta(\zeta, \bar{\zeta})},$$
$$\eta(\zeta, \bar{\zeta}) = -i\Delta^{-1}\left[U_1^1\left(\frac{\partial}{\partial\bar{\zeta}}U_1^2 - \frac{\partial}{\partial\zeta}U_2^2\right) - U_1^2\left(\frac{\partial}{\partial\zeta}U_2^1 - \frac{\partial}{\partial\bar{\zeta}}U_1^1\right)\right], \tag{1.14}$$
$$\Delta = U_1^2U_2^1 - U_1^1U_2^2.$$

Using the representation (1.12) it can be confirmed that the integrability condition $\partial\eta/\partial\bar{\zeta} = \partial\bar{\eta}/\partial\zeta$ for the system (1.14) agrees with the strain compatibility equation (1.9) which components of the tensor **G** satisfy by assumptions.

If the value $\chi_0 = \chi(\zeta_0, \bar{\zeta}_0)$ of the angle of rotation is given at a certain point M_0 with the complex coordinates ζ_0, $\bar{\zeta}_0$, then the field of rotations in the case of a simply connected domain is determined uniquely from the system (1.14). After having determined $\chi(\zeta, \bar{\zeta})$ the function $z(\zeta, \bar{\zeta})$ is found uniquely by integrating the system (1.11) for the given value $z_0 = z(\zeta_0, \bar{\zeta}_0)$.

2. We assume that the material body in the reference configuration (undeformed state) occupies a doubly connected domain. This domain can be transformed into a simply connected domain by drawing a slit (partition) along a certain curve σ. Let us examine the path of integration consisting of a curve connecting the points M_0 and M and not intersecting the partition σ, and a closed contour not shrinkable to a point that consists of n complete turns in a positive direction. The solution of (1.14) in a doubly connected domain is multivalued and has the form

$$\chi = \chi_* + nK, \quad \chi_* = \chi_0 + \int_{M_0}^{M} \eta d\zeta + \bar{\eta} d\bar{\zeta}; \tag{2.1}$$

$$K = \oint \eta d\zeta + \bar{\eta} d\bar{\zeta}. \tag{2.2}$$

A multivalued expression for z determining the location of particles of the medium in the deformed state follows from (1.11) and (2.1):

$$z = z_0 + e^{inK} \int_{M_0}^{M} e^{i\chi_*}(U_1^1 d\zeta + U_2^1 d\bar{\zeta}) + \left(1 + e^{iK} + \ldots + e^{i(n-1)K}\right) \oint e^{i\chi_*}(U_1^1 d\zeta + U_2^1 d\bar{\zeta}). \tag{2.3}$$

After transformation of the domain into a single-valued domain by drawing the slit σ the ambiguity of the functions χ and z is eliminated but the limit values of these functions on opposite sides of the slit do not agree. There results from (2.2) and (2.3) that the limit values on different sides of the partition are connected by the relations

$$\chi_+ - \chi_- = K, \; z_+ = e^{iK} z_- + \beta; \tag{2.4}$$

$$\beta = \oint \exp\left[i\chi_0 + i\int_{M_0}^{M} (\eta d\zeta + \bar{\eta} d\bar{\zeta})\right](U_1^1 d\zeta' + U_2^1 d\bar{\zeta}') + z_0(1 - e^{iK}). \tag{2.5}$$

Traversal around the closed contour in (2.2), (2.3), and (2.5) is performed from the side of the slit σ marked by a minus sign to the side marked by a plus. Moreover, the closed contour in (2.3) and (2.5) should start and terminate at the point M_0.

Formula (2.4) shows that the position of one edge of the slit in the deformed state is different from the position of the other by a finite plane displacement of an absolutely solid body, where the real constant K is the angle of finite rotation and the complex constant β determines the relative translational displacement of the slit edges. The actual realization of such a strain state generally requires removal or addition of material. The relationship (2.4) expresses the generalization of the Weingarten theorem [1, 9] of classical elasticity theory to the nonlinear case.

A formula for the jump of the displacement vector **u** results from (2.4)

$$\mathbf{u}_+ - \mathbf{u}_- = \left(1 + \frac{1}{4}\mathbf{q}\cdot\mathbf{q}\right)^{-1} \mathbf{q} \times \left(\mathbf{R}_- + \frac{1}{2}\mathbf{q} \times \mathbf{R}_-\right) + \mathbf{b}, \tag{2.6}$$
$$\mathbf{u} = \mathbf{R} - \mathbf{r}, \; \mathbf{R} = X_n \mathbf{e}_n, \; \mathbf{r} = x_n \mathbf{e}_n;$$

$$\mathbf{q}=2\,\mathrm{tg}\frac{K}{2}\mathbf{e}_3,\quad \mathbf{b}=\mathrm{Re}\,\beta\mathbf{e}_1+\mathrm{Im}\,\beta\mathbf{e}_2. \tag{2.7}$$

For a continuous strain tensor field satisfying the compatibility equation and in the presence of a doubly connected domain of the displacement jump corresponding to a rigid displacement on the slit, one speaks in linear elasticity theory about Volterra dislocations or distortion [1, 9-11]. Analogously, in a doubly connected nonlinearly elastic body a Volterra dislocation, or isolated defect, is contained if the constant vectors **b** and **q** are not simultaneously zero. As in linear elasticity theory [9], we call the isolated defect characteristics **b** and **q** the Burgers and Frank vectors, respectively. Formulas (2.2), (2.5), and (2.7) yield an expression for the isolated defect characteristics in the plane case in terms of the finite strain tensor field.

A case is possible when the domain occupied by the material body in the deformed state is not simply connected, and it is required to determine the reference configuration of the body according to the Almansey strain tensor field as a function of Euler coordinates $\mathbf{E}'=\frac{1}{2}\times\mathbf{I}-\frac{1}{2}(\mathbf{C}^T\cdot\mathbf{C})^{-1}$. This case is considered by a method analogous to that elucidated above, with the sole difference that the reference and deformed configurations exchange roles. The displacement vector jump on the slit of a non-simply connected domain is determined in this case by the relationship $\mathbf{u}_+-\mathbf{u}_-=-\left(1+\frac{1}{4}\mathbf{q}'\cdot\mathbf{q}'\right)^{-1}\mathbf{q}'\times\left(\mathbf{r}_-+\frac{1}{2}\mathbf{q}'\times\mathbf{r}_-\right)+\mathbf{b}$.

3. Let q^n be curvilinear coordinates in the reference configuration (Lagrange coordinates). The Lagrange vector basis of the deformed configuration is found from $\mathbf{R}_n=\frac{\partial X_m}{\partial q^n}\mathbf{e}_m$, $\mathbf{R}^k\cdot\mathbf{R}_n=\delta_n^k$.

The equilibrium equations in the absence of mass forces can be written in the form [11, p. 38]

$$\frac{\partial}{\partial q^n}(\sqrt{D}t^{nm})+\Gamma_{nk}^m\sqrt{D}t^{nk}=0, \tag{3.1}$$

where $t^{nm}=\mathbf{R}^n\cdot\mathbf{T}\cdot\mathbf{R}^m$; $\Gamma_{nk}^m=\frac{1}{2}G^{ms}\left(\frac{\partial G_{sn}}{\partial q^k}+\frac{\partial G_{sk}}{\partial q^n}-\frac{\partial G_{nk}}{\partial q^s}\right)$; $G_{nk}=\mathbf{R}_n\cdot\mathbf{R}_k$; $D=\det\|G_{nk}\|$; and T is the Cauchy stress tensor. According to the governing relationships for an elastic material [11, p. 360]

$$t^{nm}=2\sqrt{d/D}\frac{\partial W}{\partial G_{nm}},\quad d=\det\|g_{nk}\|, \tag{3.2}$$
$$g_{nk}=\mathbf{r}_n\cdot\mathbf{r}_k,\quad \mathbf{r}_n=\frac{\partial x_m}{\partial q^n}\mathbf{e}_m$$

(W is the specific potential strain energy).

Appending the compatibility equations (1.8) to (3.1) and (3.2), wherein x_α must be replaced by q^α, we obtain a complete system of equations for $G_{\alpha\beta}$ in the plane problem of nonlinear elasticity theory. In the case of a doubly connected domain the relations (2.2) and (2.5) that give the Volterra dislocation parameters, must be added to the equations mentioned.

As regards the boundary conditions in the stresses, then as is known [2, pp. 131, 132], in nonlinear elasticity theory we are limited to consideration of "dead" and "following" loadings. For the "dead" loading the boundary conditions have the form $\sqrt{D/d}\,n_s t^{sm}\mathbf{R}_m=\mathbf{f}^0$. Here $\mathbf{n}=n_s\mathbf{r}^s$ is the normal to the body surface in the reference configuration, and $\mathbf{f}^0$ is the surface force given on this same surface. For the most typical case of a "following" loading, hydrostatic pressure, we write the boundary conditions in the form $n_s t^{sm}G_{mk}=-pn_k$ (p is the pressure intensity). Therefore, for the case of a "following" pressure the boundary conditions are written successfully directly in the components of the strain measure **G**, for the "dead" loading it is already necessary to know the rotation field determined in terms of **G** (in the plane strain case) by means of (2.1).

Let us apply the relationships obtained to solve the problem of a defect in an elastic ring $a\leqslant r\leqslant b$.

We will seek the solution of (3.1) in the form

$$\mathbf{G}=G_{11}(r)\mathbf{r}^1\mathbf{r}^1+G_{22}(r)\mathbf{r}^2\mathbf{r}^2+\mathbf{r}^3\mathbf{r}^3. \tag{3.3}$$

Here $\mathbf{r}^n$ is the vector basis reciprocal to the Lagrange basis of the reference configuration $\mathbf{r}_n$, corresponding to the cylindrical coordinates $q^1 = r$, $q^2 = \varphi$, $q^3 = z$.

The strain compatibility equation (1.8) takes the following form in this case

$$G_{11}G_{22}\frac{d^2G_{22}}{dr^2} - \frac{1}{2}\frac{dG_{22}}{dr}\left(G_{22}\frac{dG_{11}}{dr} + G_{11}\frac{dG_{22}}{dr}\right) = 0. \quad (3.4)$$

Using the positive-definiteness of the tensor $\mathbf{G}$, we introduce the positive functions A(r) and B(r) such that

$$G_{11}(r) = A^2(r),\ G_{22}(r) = B^2(r). \quad (3.5)$$

By using these functions we write (3.4) as $A\frac{d^2B}{dr^2} - \frac{dB}{dr}\frac{dA}{dr} = 0$. Integrating this relationship and denoting the constant of integration in terms of $\ln \varkappa (\varkappa > 0)$, we obtain $|dB/dr| = \varkappa A$. The absolute value sign can be omitted in this relationship but then $\varkappa$ is considered arbitrary, either positive or negative. The case $\varkappa < 0$ (or $dB/dr < 0$) corresponds to strain accompanied by rotation of the ring inside out, and will not be considered here. Consequently, by omitting the absolute value sign henceforth, we consider $\varkappa > 0$:

$$dB/dr = \varkappa A. \quad (3.6)$$

In our case the relationship (2.2) for the Frank angle takes the form

$$K = \int_0^{2\pi}\left(\frac{r}{U_{11}}\frac{dU_{22}}{dr} + \frac{U_{22}}{U_{11}} - 1\right)d\varphi. \quad (3.7)$$

Here U_{11} and U_{22} are components of the distortion measure $\mathbf{U}$ in the orthonormal basis of the cylindrical coordinates. Taking (3.5) into account $U_{11}(r) = A(r)$ and $U_{22}(r) = r^{-1}B(r)$. Taking (3.6) into account (3.7) is transformed into $K = 2\pi(\varkappa - 1)$.

Therefore, according to a given Frank angle K is determined by the constant $\varkappa$:

$$\varkappa = (2\pi + K)/2\pi. \quad (3.8)$$

In this case the field of rotations is $\chi_* = (\varkappa - 1)\varphi$. We consider here that the slit is drawn along the line $\varphi = 0$ and we set $\chi(M_0) = 0$ for the point M_0 lying on this slit.

Transforming (2.5) for β, we have the relationship $\beta = (1 - e^{iK})(z_0 - (1/\varkappa)B(r_0))$, which shows that the problem of a wedge disclination (rotational dislocation) is solved successfully by the representation (3.3), while such a representation is still inadequate to the solution of the problem of a translational dislocation. Indeed, by setting $K = 0$ (no rotational dislocation), we obtain that $\beta = 0$ also, i.e., there is generally no defect in the body.

When the defect characteristics $K = 0$, $\beta \neq 0$ are given no solution of a problem of the form (3.3) exists.

We study the stress–strain state of a ring with disclinations for a semilinear material [2, 11]. The expression for W has the form $W = (1/2)\lambda\, \mathrm{tr}^2\,(\mathbf{U} - \mathbf{I}) + \mu\, \mathrm{tr}\,((\mathbf{U} - \mathbf{I})^2)$ (λ and μ are material constants).

The expressions for the stress tensor components (3.2) are the following in this case when the representations (3.3) and (3.5) are taken into account:

$$t^{11} = rA^{-2}B^{-1}[\lambda(A + r^{-1}B) - 2(\lambda + \mu)] + 2\mu rA^{-1}B^{-1},$$
$$t^{22} = A^{-1}B^{-2}[\lambda(A + r^{-1}B) - 2(\lambda + \mu)] + 2\mu r^{-1}A^{-1}B^{-1},$$
$$t^{33} = \lambda rA^{-1}B^{-1}(A + r^{-1}B - 2).$$

Substituting these expressions into the equilibrium equations (3.1) we find that the second and third are satisfied identically while the first takes the following form after manipulation

$$(\lambda + 2\mu)\left(A\frac{dA}{dr} + r^{-1}A^2 - r^{-2}B\frac{dB}{dr}\right) + 2r^{-1}(\lambda + \mu)\left(\frac{\partial B}{dr} - A\right) = 0. \quad (3.9)$$

Solving the system (3.6) and (3.9), we write

$$A(r) = C_1r^{\varkappa-1} + C_2r^{-\varkappa-1} + \frac{1}{(1+\varkappa)(1-\nu)},$$

$$B(r) = C_1 r^{\varkappa} - C_2 r^{-\varkappa} + \frac{\varkappa}{(1+\varkappa)(1-\nu)} r, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Here C_1 and C_2 are arbitrary constants and ν is the Poisson ratio. The constant $\varkappa$ is expressed in terms of the Frank angle by the relationship (3.8).

We use the boundary conditions to determine the constants. We consider the ring surfaces ($r = a$ and $r = b$) load free. In this case we write the boundary conditions as $(\lambda + 2\mu)A(r) + r^{-1}\lambda B(r) - 2(\lambda + \mu) = 0$, $r = a, b$. We find expressions for the constants

$$C_1 = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa}\frac{b^{\varkappa+1} - a^{\varkappa+1}}{b^{2\varkappa} - a^{2\varkappa}},$$
$$C_2 = \frac{1}{1-\nu}\frac{\varkappa}{1+\varkappa}\frac{b^{\varkappa-1} - a^{\varkappa-1}}{b^{2\varkappa} - a^{2\varkappa}} a^{\varkappa+1} b^{\varkappa+1}. \tag{3.10}$$

Let us consider the case of a continuous disc ($a = 0$). From (3.10) $C_1 = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa} \times b^{1-\varkappa}$, $C_2 = 0$. The expressions for the strains take the form

$$U_{11} = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa}\rho^{\varkappa-1} + \frac{1}{(1+\varkappa)(1-\nu)},$$
$$U_{22} = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa}\rho^{\varkappa-1} + \frac{\varkappa}{(1+\varkappa)(1-\nu)}, \quad \rho = \frac{r}{a}.$$

It is seen from these formulas that for $\varkappa > 1$ the strain field has no singularities on the disclination axis (in the neighborhood of $\rho = 0$). For $\varkappa < 1$ a singularity of order $\rho^{\varkappa-1}$ exists for this field.

Analysis of the relationship (3.8) shows that the case $\varkappa < 1$ corresponds to the introduction of a wedge with apex angle K into a slit of the ring, and $\varkappa > 1$ corresponds to strain that occurs after removal of part of the material in the form of a sector from the ring with subsequent connection of the edges.

We obtain expressions for the principal stresses σ_1, σ_2, and σ_3 in terms of the components t^{mn} in the case of a continuous disc by means of the formulas $\sigma_1 = t^{11}G_{11}$, $\sigma_2 = t^{22}G_{22}$, $\sigma_3 = t^{33}$ in the form

$$\sigma_1 = \frac{2\mu(\rho^{\varkappa-1} - 1)}{(1-2\nu)\rho^{\varkappa-1} + 1}, \quad \sigma_2 = \frac{2\mu(\varkappa\rho^{\varkappa-1} - 1)}{(1-2\nu)\varkappa\rho^{\varkappa-1} + 1},$$
$$\sigma_3 = \frac{2\mu\nu(1-\nu)(1+\varkappa)(2\varkappa\rho^{\varkappa-1} - \varkappa - 1)}{\varkappa[(1-2\nu)\varkappa\rho^{\varkappa-1} + 1][(1-2\nu)\rho^{\varkappa-1} + 1]}.$$

It is seen that the principal stresses have no singularities on the disclination axis.

The results obtained about the behavior of the stress and strain near a defect axis with a rigorous accounting of the geometric nonlinearity differ qualitatively from the results of linear elasticity theory [9, 11], according to which the strain and stress fields have a logarithmic singularity on the disclination axis.

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